# SHORT ORDER MATHEMATICS

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ABSTRACT. This note introduces some short order food-related theorems from topology, the Pancake Theorem and the Ham Sandwich Theorem.

## 1. Introduction

The Pancake Theorem and the Ham Sandwich Theorem show that there is a way to divide evenly two pancakes or a three-layer ham sandwich.

## 2. The Borsuk-Ulam Theorem

This theorem is needed to prove the "short order" theorems.

**Definition 1.** *Consider the n-sphere $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$. The points $x = (x_1, x_2, \ldots, x_{n+1})$ and $-x = (-x_1, -x_2, \ldots, -x_{n+1})$ in $\mathbb{S}^n$ are called* antipodal.

**Theorem 1.** *If $f$ is a continuous function from the circle $\mathbb{S}^1$ to a subset of $\mathbb{R}$, then there is a pair of antipodal points $x$ and $-x$ on $\mathbb{S}^1$ with $f(x) = f(-x)$. In other words there is a set of antipodal points where $f$ has the same value.*

*Proof.* Assume $f(x) \neq f(-x)$ for all $x \in \mathbb{S}^1$. Define a new function $g$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Then g is continuous, since $f$ is continuous and we have not divided by zero. Note that

$$g(-x) = \frac{f(-x) - f(x)}{|f(-x) - f(x)|} = -g(x) \tag{1}$$

Thus, if $g(x) = +1$, then from (1) $g(-x) = -1$, and vice versa. We have constructed a continuous function $g$ from the connected space $\mathbb{S}^1$ onto the set $\{-1, 1\}$ which is not connected. This gives a contradiction unless there is, after all, a point $x$ where $f(x) = f(-x)$. □

**Corollary 1.** *At any given moment, there are two points on the equator exactly opposite each other with the same temperature.*

*Proof.* The equator can be considered to be the circle $\mathbb{S}^1$ by choosing the appropriate units. Let $f$ be the temperature function and use Theorem 1. □

*Date*: July 17, 2001.

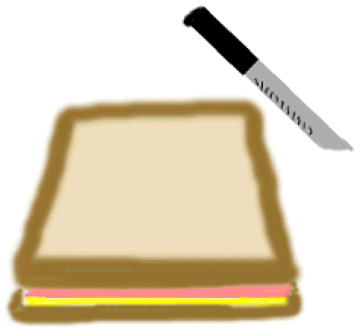

Figure 1. Ham sandwich (drawing courtesy of Francis Su [3])

## 3. The Pancake Theorem

The following theorem implies that there is a single knife cut that will divide two objects of uniform thickness. If you are sharing your pancake breakfast with a friend, this shows that there *is* a way to divide it evenly but it doesn't tell *how* to do it, which may cause some problems with your friend.

**Theorem 2.** *Let $A$ and $B$ be bounded connected open subsets of $\mathbb{R}^2$, which may overlap. There is a line which divides each region in half.*

*Proof.* Intuitively, think of two pancakes on a plate. Slowly pass a knife starting from the left side of the plate where there are no pancakes, over both pancakes to the right side of the plate. There is a point where, if you had brought the knife down, the pancakes would be divided evenly. For a formal proof see [1]. $\square$

## 4. The Ham Sandwich Theorem

Now we come to the Ham Sandwich Theorem, which says that three objects, such as a slice of ham and two pieces of bread, may be evenly divided with a single knife cut. In a talk at MIT [2] Donald Knuth showed that the result holds for a variety of sandwiches, including cheese, strawberry jam, and even knuckle sandwiches.

**Theorem 3.** *Let $A$, $B$, and $C$ be compact connected subsets of $\mathbb{R}^3$. There is a plane which divides each region exactly in half.*

*Proof.* This is intuitively similar to the Pancake Theorem (see Figure 1) but the proof is more involved. For a formal proof see [1]. $\square$

## References

1. L. Christine Kinsey, *Topology of surfaces*, Springer, 1993.
2. Donald E. Knuth, *On generalizations of the ham-sandwich theorem*, `http://www.csail.mit.edu/events/eventcalendar/series_exp.php?show=event&id=420`, April 1 2005, CSAIL talk at MIT.
3. Francis E. Su, *Ham sandwich theorem*, `http://www.math.hmc.edu/funfacts/ffiles/20001.7.shtml`, 2007, Mudd Math Fun Facts.

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